Lecture 7
Measurement Using a Single Camera

Lin ZHANG, PhD
School of Software Engineering
Tongji University
Fall 2016
If I have an image containing a coin, can you tell me the diameter of that coin?
Contents

• Foundations of Projective Geometry
• Matrix Differentiation
• Lagrange Multiplier
• Least-squares for Linear Systems
• What is Camera Calibration
• Single Camera Calibration
• Bird’s-eye-view Generation
Vector operations

Vector representation
\[ \vec{a} = x \hat{i} + y \hat{j} + z \hat{k} = \{x, y, z\} \]

Length (or norm) of a vector
\[ |\vec{a}| = \sqrt{x^2 + y^2 + z^2} \]

Normalized vector (unit vector)
\[ \vec{a} = \{ \frac{x}{|\vec{a}|}, \frac{y}{|\vec{a}|}, \frac{z}{|\vec{a}|} \} \]

We say \( \vec{a} = \mathbf{0} \), if and only if \( x = 0, y = 0, z = 0 \)
Vector operations

if \( \vec{a} = (x_1, y_1, z_1), \vec{b} = (x_2, y_2, z_2), \)
then \( \vec{a} \pm \vec{b} = (x_1 \pm x_2, y_1 \pm y_2, z_1 \pm z_2), \)

Dot product (inner product)

\[ \vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta = x_1 x_2 + y_1 y_2 + z_1 z_2 \]

Laws of dot product:

\[ \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}, \vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \]

Theorem

\[ \vec{a} \cdot \vec{b} = 0 \iff \vec{a} \perp \vec{b} \quad \text{(why?)} \]
Vector operations

Cross product

\[ \vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} i + \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} j + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} k \]
Vector operations

Cross product

\[ \vec{c} = \vec{a} \times \vec{b} \] is also a vector, whose direction is determined by the right-hand law and

\[ \vec{c} \perp \vec{a}, \vec{c} \perp \vec{b} \]

\[ |\vec{c}| = |\vec{a}| |\vec{b}| \sin \theta \]

\[ \vec{c} \] represents the oriented area of the parallelogram taking and \( \vec{a} \) as two sides \( \vec{b} \) (easy to prove)

\[ \vec{r}_1 \times \vec{r}_2 = -\vec{r}_2 \times \vec{r}_1 \] (why?)
Vector operations

Cross product

Theorem

\[ \vec{a} \parallel \vec{b} \iff \vec{a} \times \vec{b} = \vec{0} \] (why?)

Theorem

\[ \vec{a} \parallel \vec{b} \iff \exists \lambda, \mu, \text{ they are not equal to zero at the same time, and } \lambda \vec{a} + \mu \vec{b} = \vec{0} \] (easy to understand)

Property

\[ \vec{r}_1 \times (\vec{r}_2 + \vec{r}_3) = \vec{r}_1 \times \vec{r}_2 + \vec{r}_1 \times \vec{r}_3 \]
Vector operations

Mixed product (scalar triple product or box product)

\[(a, b, c) = (a \times b) \cdot c = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \]

Geometric Interpretation: it is the (signed) volume of the parallelepiped defined by the three vectors given
Vector operations

Mixed product (scalar triple product or box product)

\[(a, b, c) = (a \times b) \cdot c = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \]

\[(a \times b) \cdot c = |a \times b| |c| \cos \alpha = |a| |b| \sin \theta \cdot |c| \cos \alpha\]
Vector operations

Mixed product (scalar triple product or box product)

\[(a, b, c) = (a \times b) \cdot c = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\]

Property:

\[(a, b, c) = (b, c, a) = (c, a, b)\]
\[(a, b, c) = -(b, a, c) = -(a, c, b)\]
Vector operations

Mixed product (scalar triple product or box product)

Theorem

\( \mathbf{a}, \mathbf{b}, \mathbf{c} \) are coplanar \( \iff (\mathbf{a}, \mathbf{b}, \mathbf{c}) = 0 \)

\( \mathbf{a}, \mathbf{b}, \mathbf{c} \) are coplanar \( \iff \exists \lambda, \mu, \nu, \) they are not equal to zero at the same time, and \( \lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{c} = 0 \)

why?
• What is homogeneous coordinate?

For a normal point \((x, y)^T\), on a plane \(\pi_0\)

Its homogenous coordinate is \(k(x, y, 1)^T\), where \(k\) can be any non-zero real number.

Homogenous coordinate for a point is not only one.

For a homogenous coordinate \((x', y', z')^T\)
we usually rewrite it as \((x' / z', y' / z', 1)^T\)
Foundations of Projective Geometry

• What is homogeneous coordinate?

For a normal point \((x, y)^T\), on a plane \(\pi_0\)

Its homogenous coordinate is \(k(x, y, 1)^T\), where \(k\) can be any non-zero real number

Converting from homogenous coordinate to inhomogeneous coordinate

\[
\begin{pmatrix}
 x' \\
 y' \\
 z'
\end{pmatrix} \Rightarrow \begin{pmatrix}
 \frac{x'}{z'} \\
 \frac{y'}{z'}
\end{pmatrix}
\]
• What is homogeneous coordinate?

Geometric interpretation

In plane $\pi_0$, in the 2D frame $(o_1 : e_1, e_2)$, one point $M : (x_0, y_0)$

Coordinate of any point on line $OM$ in the frame $(o : e_1, e_2, e_3)$ is the homogeneous coordinate of $M$

These points can be represented as $k(x_0, y_0, 1)^T$
• What is homogeneous coordinate?

Geometric interpretation

In plane $\pi_0$, in the 2D frame $(o_1 : e_1, e_2)$, one point $M : (x_0, y_0)$

Coordinate of any point on line $OM$ in the frame $(o : e_1, e_2, e_3)$ is the homogeneous coordinate of $M$

These points can be represented as $k(x_0, y_0, 1)^T$

How about a line passing through $O$ and parallel to $\pi_0$?
- What is homogeneous coordinate?

Geometric interpretation

How about a line passing through $O$ and parallel to $\pi_0$?

Consider a line passing through $O$ and $M(x_0, y_0, 0)^T$

We define: it meets $\pi_0$ at an infinity point, and also the homogeneous coordinate of such a point can be represented as points on $OM$.

So, infinity point has the form $(kx_0, ky_0, 0)^T$.
What is homogeneous coordinate?

**Normal case:**

- Line: \( k(x_0, y_0, 1) \)
- A normal point: \( (x_0, y_0) \) on the plane \( \pi_0 \)

The homogeneous coordinate of this normal point is \( k(x_0, y_0, 1) \)

**Abnormal case:**

- Line: \( (kx_0, ky_0, 0) \)
- Define: it meets \( \pi_0 \) at an infinity point

The homogeneous coordinate of this infinity point is \( k(x_0, y_0, 0) \)
Foundations of Projective Geometry

• What is homogeneous coordinate?

Geometric interpretation

How about a line passing through $O$ and parallel to $\pi_0$?

One infinity point determines an orientation

We define: all infinity points on $\pi_0$ comprise an **infinity line**

In fact, plane $Oe_1e_2$ meets $\pi_0$ at the infinity line

Homogeneous equation of the infinity line is $0x + 0y + 1z = 0$
Foundations of Projective Geometry

\[ \pi_0 + \text{infinity line} = \text{Projective plane} \]

Properties of a projective plane

- Two points determine a line; two lines determine a point (the second claim is not correct in the normal Euclidean plane)
- Two parallel lines intersect at an infinity point; that means one infinity point corresponds to a specific orientation
- Two parallel planes intersect at the infinity line
Foundations of Projective Geometry

Lin ZHANG, SSE, 2016
JingHu High-speed railway: rails will “meet” at the vanishing point
Foundations of Projective Geometry

• Lines in the homogeneous coordinate

On a projective plane, please determine the line passing two points \( \mathbf{x} = (x_1, y_1, z_1)^T \), \( \mathbf{x}' = (x_2, y_2, z_2)^T \)

\[ \mathbf{ox}, \mathbf{ox}' \text{ determine two lines} \]

\[ \mathbf{xx}' \text{ actually is the intersection between } \mathbf{oxx}' \text{ and } \pi_0 \]
Foundations of Projective Geometry

• Lines in the homogeneous coordinate

On a projective plane, please determine the line passing two points \( \mathbf{x} = (x_1, y_1, z_1)^T, \mathbf{x}' = (x_2, y_2, z_2)^T \)

Thus, \( M(x, y, z) \) locates on \( \mathbf{xx}' \)

\( \iff \) \( oM \) resides on the plane \( o\mathbf{xx}' \)

\( \iff \) \( o\mathbf{x}, oM, o\mathbf{x}' \) are coplanar

\[
\begin{vmatrix}
    x & y & z \\
    x_1 & y_1 & z_1 \\
    x_2 & y_2 & z_2 \\
\end{vmatrix} = 0
\]
Lines in the homogeneous coordinate

On a projective plane, please determine the line passing two points \( \mathbf{x} = (x_1, y_1, z_1)^T \), \( \mathbf{x}' = (x_2, y_2, z_2)^T \)

\[
\begin{vmatrix}
x & y & z \\
x_1 & y_1 & z_1 \\
x_2 & y_2 & z_2
\end{vmatrix} = 0 \Leftrightarrow 
\begin{vmatrix}
y_1 & z_1 \\
y_2 & z_2
\end{vmatrix} x + \begin{vmatrix}
z_1 & x_1 \\
z_2 & x_2
\end{vmatrix} y + \begin{vmatrix}
x_1 & y_1 \\
x_2 & y_2
\end{vmatrix} z = 0
\]

\[
\begin{pmatrix}
y_1 & z_1 \\
y_2 & z_2
\end{pmatrix}, \begin{pmatrix}
z_1 & x_1 \\
z_2 & x_2
\end{pmatrix}, \begin{pmatrix}
x_1 & y_1 \\
x_2 & y_2
\end{pmatrix}
\]

Homogeneous coordinate of the line

Homogeneous coordinate of the infinity line is \((0,0,1)^T\)
Foundations of Projective Geometry

• Lines in the homogeneous coordinate

On a projective plane, please determine the line passing two points \( \mathbf{x} = (x_1, y_1, z_1)^T \), \( \mathbf{x}' = (x_2, y_2, z_2)^T \)

\[
\begin{vmatrix}
  x & y & z \\
  x_1 & y_1 & z_1 \\
  x_2 & y_2 & z_2 \\
\end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix}
  y_1 & z_1 \\
  y_2 & z_2 \\
\end{vmatrix} x + \begin{vmatrix}
  z_1 & x_1 \\
  z_2 & x_2 \\
\end{vmatrix} y + \begin{vmatrix}
  x_1 & y_1 \\
  x_2 & y_2 \\
\end{vmatrix} z = 0
\]

\[
\left( \begin{vmatrix}
  y_1 & z_1 \\
  y_2 & z_2 \\
\end{vmatrix}, \begin{vmatrix}
  z_1 & x_1 \\
  z_2 & x_2 \\
\end{vmatrix}, \begin{vmatrix}
  x_1 & y_1 \\
  x_2 & y_2 \\
\end{vmatrix} \right)^T
\]

Theorem

On the projective plane, the line passing two points \( \mathbf{x}, \mathbf{x}' \) is

\[
\mathbf{l} = \mathbf{x} \times \mathbf{x}'
\]
Foundations of Projective Geometry

• Lines in the homogeneous coordinate

A point $\mathbf{x} = (x_0, y_0, z_0)^T$ is on the line $\mathbf{l} = (a, b, c)^T$

$\iff \mathbf{x}^T \mathbf{l} = 0$ (It is $\mathbf{x} \cdot \mathbf{l} = 0$)
Foundations of Projective Geometry

• Lines in the homogeneous coordinate

Theorem: On the projective plane, the intersection of two lines $l, l'$ is the point $x = l \times l'$

Proof: Two lines $a_1x + b_1y + c_1z = 0, a_2x + b_2y + c_2z = 0$

Inhomogeneous form

$\begin{cases} a_1X + b_1Y + c_1 = 0 \\ a_2X + b_2Y + c_2 = 0 \end{cases}$

$X = \begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}, \quad Y = \begin{vmatrix} a_1 - c_1 \\ a_2 - c_2 \end{vmatrix}$
Foundations of Projective Geometry

• Lines in the homogeneous coordinate

Theorem: On the projective plane, the intersection of two lines $l, l'$ is the point $x = l \times l'$

Homogenous form of the cross point is

$$x = k \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \\ a_1 b_1 \\ a_2 b_2 \end{pmatrix} \quad \text{let} \quad k = \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \end{pmatrix}$$

$$x = \begin{pmatrix} -c_1 b_1 \\ -c_2 b_2 \\ a_1 c_1 \\ a_2 c_2 \end{pmatrix} \quad \implies \quad x = \begin{pmatrix} b_1 c_1 \\ b_2 c_2 \\ c_1 a_1 \\ c_2 a_2 \end{pmatrix}$$

Lin ZHANG, SSE, 2016
Foundations of Projective Geometry

• Lines in the homogeneous coordinate

Theorem: On the projective plane, the intersection of two lines \( \ell, \ell' \) is the point \( x = \ell \times \ell' \)

Example: find the cross point of the lines \( x = 1, y = 1 \)

Homogeneous form

\[
\begin{align*}
1x_1 + 0x_2 + (-1)x_3 &= 0 \\
0x_1 + 1x_2 + (-1)x_3 &= 0
\end{align*}
\]

Homogeneous coordinates of the two lines are \((1, 0, -1)^T, (0, 1, -1)^T\)

Cross point is

\[(1, 0, -1)^T \times (0, 1, -1)^T = (1, 1, 1)\]
Foundations of Projective Geometry

- Lines in the homogeneous coordinate

Theorem: On the projective plane, the intersection of two lines \( l, l' \) is the point \( x = l \times l' \)

Example: find the cross point of the lines \( x = 1, x = 2 \)

Homogeneous form

\[
\begin{align*}
1x_1 + 0x_2 + (-1)x_3 &= 0 \\
1x_1 + 0x_2 + (-2)x_3 &= 0
\end{align*}
\]

Homogeneous coordinates of the two lines are \( (1, 0, -1)^T, (1, 0, -2)^T \)

Cross point is

\[
(1, 0, -1)^T \times (1, 0, -2)^T = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 1 & 0 & -2 \end{vmatrix} = (0, 1, 0)
\]
Duality

In projective geometry, lines and points can swap their positions

\[ x^T l = 0 \quad How \ to \ interpret? \]

If \( x \) is a variable, it represents the points lying on the line \( l \);
If \( l \) is a variable, it represents the lines passing a fixed point \( x \)

The line passing two points \( x, x' \) is \( l = x \times x' \)
The cross point of two lines \( l, l' \) is \( x = l \times l' \)

**Duality Principle:** To any theorem of projective geometry, there corresponds a dual theorem, which may be derived by interchanging the roles of points and lines in the original theorem
Contents

• Foundations of Projective Geometry
• Matrix Differentiation
• Lagrange Multiplier
• Least-squares for Linear Systems
• What is Camera Calibration
• Single Camera Calibration
• Bird’s-eye-view Generation
Matrix differentiation

• Function is a vector and the variable is a scalar

\[ f(t) = \left[ f_1(t), f_2(t), \ldots, f_n(t) \right]^T \]

Definition

\[ \frac{df}{dt} = \left[ \frac{df_1}{dt}, \frac{df_2}{dt}, \ldots, \frac{df_n}{dt} \right]^T \]
Matrix differentiation

- Function is a matrix and the variable is a scalar

\[
f(t) = \begin{bmatrix}
f_{11}(t) & f_{12}(t) & \ldots & f_{1m}(t) \\
f_{21}(t) & f_{22}(t) & \ldots & f_{2m}(t) \\
\vdots & \vdots & \ddots & \vdots \\
f_{n1}(t) & f_{n2}(t) & \ldots & f_{nm}(t)
\end{bmatrix} = \begin{bmatrix} f_{ij}(t) \end{bmatrix}_{n \times m}
\]

Definition

\[
\frac{df}{dt} = \begin{bmatrix}
\frac{df_{11}(t)}{dt} & \frac{df_{12}(t)}{dt} & \ldots & \frac{df_{1m}(t)}{dt} \\
\frac{df_{21}(t)}{dt} & \frac{df_{22}(t)}{dt} & \ldots & \frac{df_{2m}(t)}{dt} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{df_{n1}(t)}{dt} & \frac{df_{n2}(t)}{dt} & \ldots & \frac{df_{nm}(t)}{dt}
\end{bmatrix} = \begin{bmatrix} \frac{df_{ij}(t)}{dt} \end{bmatrix}_{n \times m}
\]
Matrix differentiation

• Function is a scalar and the variable is a vector

\[ f(x), \quad x = (x_1, x_2, \ldots, x_n)^T \]

Definition

\[ \frac{df}{dx} = \left[ \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right]^T \]

In a similar way,

\[ f(x), \quad x = (x_1, x_2, \ldots, x_n) \]

\[ \frac{df}{dx} = \left[ \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right] \]
Matrix differentiation

- Function is a vector and the variable is a vector
  \[ \mathbf{x} = [x_1, x_2, \ldots, x_n]^T, \mathbf{y} = [y_1(\mathbf{x}), y_2(\mathbf{x}), \ldots, y_m(\mathbf{x})]^T \]

Definition

\[
\frac{d\mathbf{y}}{d\mathbf{x}}^T = 
\begin{bmatrix}
\frac{\partial y_1(\mathbf{x})}{\partial x_1}, \frac{\partial y_1(\mathbf{x})}{\partial x_2}, \ldots, \frac{\partial y_1(\mathbf{x})}{\partial x_n} \\
\frac{\partial y_2(\mathbf{x})}{\partial x_1}, \frac{\partial y_2(\mathbf{x})}{\partial x_2}, \ldots, \frac{\partial y_2(\mathbf{x})}{\partial x_n} \\
\vdots \\
\frac{\partial y_m(\mathbf{x})}{\partial x_1}, \frac{\partial y_m(\mathbf{x})}{\partial x_2}, \ldots, \frac{\partial y_m(\mathbf{x})}{\partial x_n}
\end{bmatrix}_{m \times n}
\]
Matrix differentiation

- Function is a vector and the variable is a vector

\[ \mathbf{x} = [x_1, x_2, \ldots, x_n]^T, \quad \mathbf{y} = [y_1(\mathbf{x}), y_2(\mathbf{x}), \ldots, y_m(\mathbf{x})]^T \]

In a similar way,

\[
\frac{\partial \mathbf{y}^T}{\partial \mathbf{x}} = \begin{bmatrix}
\frac{\partial y_1(\mathbf{x})}{\partial x_1}, & \frac{\partial y_2(\mathbf{x})}{\partial x_1}, & \cdots, & \frac{\partial y_m(\mathbf{x})}{\partial x_1} \\
\frac{\partial y_1(\mathbf{x})}{\partial x_2}, & \frac{\partial y_2(\mathbf{x})}{\partial x_2}, & \cdots, & \frac{\partial y_m(\mathbf{x})}{\partial x_2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial y_1(\mathbf{x})}{\partial x_n}, & \frac{\partial y_2(\mathbf{x})}{\partial x_n}, & \cdots, & \frac{\partial y_m(\mathbf{x})}{\partial x_n}
\end{bmatrix}_{n \times m}
\]
Matrix differentiation

• Function is a vector and the variable is a vector

Example:

\[ y = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad y_1(x) = x_1^2 - x_2, \quad y_2(x) = x_3^2 + 3x_2 \]

\[ \frac{dy^T}{dx} = \begin{bmatrix} \frac{\partial y_1(x)}{\partial x_1} & \frac{\partial y_2(x)}{\partial x_1} \\ \frac{\partial y_1(x)}{\partial x_2} & \frac{\partial y_2(x)}{\partial x_2} \\ \frac{\partial y_1(x)}{\partial x_3} & \frac{\partial y_2(x)}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 2x_1 & 0 \\ -1 & 3 \\ 0 & 2x_3 \end{bmatrix} \]
Matrix differentiation

- Function is a scalar and the variable is a matrix

\[ f(X), X \in \mathbb{R}^{m\times n} \]

Definition

\[
\frac{df(X)}{dX} = \begin{bmatrix}
\frac{\partial f}{\partial x_{11}} & \frac{\partial f}{\partial x_{12}} & \ldots & \frac{\partial f}{\partial x_{1n}} \\
\frac{\partial f}{\partial x_{m1}} & \frac{\partial f}{\partial x_{m2}} & \ldots & \frac{\partial f}{\partial x_{mn}}
\end{bmatrix}
\]
Matrix differentiation

• Useful results

(1) \( x, a \in \mathbb{R}^{n \times 1} \)

Then,

\[
\frac{da^T x}{dx} = a, \quad \frac{dx^T a}{dx} = a
\]

How to prove?
Matrix differentiation

• Useful results

(2) \( A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^{n \times 1} \) Then, \( \frac{dAx}{dx^T} = A \)

(3) \( A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^{n \times 1} \) Then, \( \frac{dx^T A^T}{dx} = A^T \)

(4) \( A \in \mathbb{R}^{n \times n}, x \in \mathbb{R}^{n \times 1} \) Then, \( \frac{dx^T Ax}{dx} = (A + A^T)x \)

(5) \( X \in \mathbb{R}^{m \times n}, a \in \mathbb{R}^{m \times 1}, b \in \mathbb{R}^{n \times 1} \) Then, \( \frac{d a^T X b}{dX} = ab^T \)

(6) \( X \in \mathbb{R}^{n \times m}, a \in \mathbb{R}^{m \times 1}, b \in \mathbb{R}^{n \times 1} \) Then, \( \frac{d a^T X^T b}{dX} = ba^T \)

(7) \( x \in \mathbb{R}^{n \times 1} \) Then, \( \frac{dx^T x}{dx} = 2x \)
Contents

• Foundations of Projective Geometry
• Matrix Differentiation
• Lagrange Multiplier
• Least-squares for Linear Systems
• What is Camera Calibration
• Single Camera Calibration
• Bird-view Generation
Lagrange multiplier

- Single-variable function

$f(x)$ is differentiable in $(a, b)$. At $x_0 \in (a, b), f(x)$ achieves an extremum

\[ \frac{df}{dx} \bigg|_{x_0} = 0 \]

- Two-variables function

$f(x, y)$ is differentiable in its domain. At $(x_0, y_0), f(x, y)$ achieves an extremum

\[ \frac{\partial f}{\partial x} \bigg|_{(x_0, y_0)} = 0, \quad \frac{\partial f}{\partial y} \bigg|_{(x_0, y_0)} = 0 \]
Lagrange multiplier

• In general case

If $\mathbf{x}_0$ is a stationary point of $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^{n \times 1}$

\[
\frac{\partial f}{\partial x_1} \bigg|_{x_0} = 0, \quad \frac{\partial f}{\partial x_2} \bigg|_{x_0} = 0, \ldots, \quad \frac{\partial f}{\partial x_n} \bigg|_{x_0} = 0
\]
Lagrange multiplier

• Lagrange multiplier is a strategy for finding the local extremum of a function subject to equality constraints.

Problem: find stationary points for \( y = f(x), \ x \in \mathbb{R}^{n \times 1} \)
under \( m \) constraints \( g_k(x) = 0, k = 1, 2, \ldots, m \)

Solution:

\[
F(x; \lambda_1, \ldots, \lambda_m) = f(x) + \sum_{k=1}^{m} \lambda_k g_k(x)
\]

If \((x_0, \lambda_{10}, \lambda_{20}, \ldots, \lambda_{m0})\) is a stationary point of \(F\), then,
\(x_0\) is a stationary point of \(f(x)\) with constraints.

Joseph-Louis Lagrange
Jan. 25, 1736~Apr.10, 1813
Lagrange multiplier

- Lagrange multiplier is a strategy for finding the local extremum of a function subject to equality constraints.

Problem: find stationary points for \( y = f(x), x \in \mathbb{R}^{n \times 1} \) under \( m \) constraints \( g_k(x) = 0, k = 1, 2, \ldots, m \)

Solution:
\[
F(x; \lambda_1, \ldots, \lambda_m) = f(x) + \sum_{k=1}^{m} \lambda_k g_k(x)
\]

\((x_0, \lambda_{10}, \ldots, \lambda_{m0})\) is a stationary point of \( F \)

\[
\frac{\partial F}{\partial x_1} = 0, \quad \frac{\partial F}{\partial x_2} = 0, \ldots, \quad \frac{\partial F}{\partial x_n} = 0, \quad \frac{\partial F}{\partial \lambda_1} = 0, \quad \frac{\partial F}{\partial \lambda_2} = 0, \ldots, \quad \frac{\partial F}{\partial \lambda_m} = 0
\]

at that point \( n + m \) equations!
Lagrange multiplier

• Example

Problem: for a given point $p_0 = (1, 0)$, among all the points lying on the line $y = x$, identify the one having the least distance to $p_0$.

The distance is

$$f(x, y) = (x - 1)^2 + (y - 0)^2$$

Now we want to find the stationary point of $f(x, y)$ under the constraint $g(x, y) = y - x = 0$

According to Lagrange multiplier method, construct another function

$$F(x, y, \lambda) = f(x) + \lambda g(x) = (x - 1)^2 + y^2 + \lambda(y - x)$$

Find the stationary point for $F(x, y, \lambda)$
Lagrange multiplier

• Example

Problem: for a given point $p_0 = (1, 0)$, among all the points lying on the line $y = x$, identify the one having the least distance to $p_0$.

\[
\begin{align*}
\frac{\partial F}{\partial x} &= 0 \\
2(x - 1) + \lambda &= 0 \\
x = 0.5
\end{align*}
\]

\[
\begin{align*}
\frac{\partial F}{\partial y} &= 0 \\
2y - \lambda &= 0 \\
y = 0.5
\end{align*}
\]

\[
\begin{align*}
\frac{\partial F}{\partial \lambda} &= 0 \\
x - y &= 0 \\
\lambda = 1
\end{align*}
\]

$(0.5, 0.5, 1)$ is a stationary point of $F(x, y, \lambda)$

$(0.5, 0.5)$ is a stationary point of $f(x, y)$ under constraints
Contents

• Foundations of Projective Geometry
• Matrix Differentiation
• Lagrange Multiplier
• Least-squares for Linear Systems
• Homography Estimation
• What is Camera Calibration
• Single Camera Calibration
• Bird-view Generation
LS for Inhomogeneous Linear System

Consider the following linear equations system

\[
\begin{align*}
7x_1 + x_2 &= 3 \\
2x_1 + x_2 &= 4
\end{align*}
\]

\[
\begin{bmatrix}
1 & 1 \\
2 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} =
\begin{bmatrix}
3 \\
4
\end{bmatrix}
\]

Matrix form: \(Ax = b\)

It can be easily solved \(x_1 = 1, x_2 = 2\)
LS for Inhomogeneous Linear System

How about the following one?

\[
\begin{align*}
    x_1 + x_2 &= 3 \\
    2x_1 + x_2 &= 4 \iff \\
    x_1 + 2x_2 &= 6
\end{align*}
\]

\[
\begin{bmatrix}
    1 & 1 \\
    2 & 1 \\
    1 & 2
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix}
=
\begin{bmatrix}
    3 \\
    4 \\
    6
\end{bmatrix}
\]

It does not have a solution!

What is the condition for a linear equation system \(Ax = b\) can be solved?

\[
\begin{align*}
    \text{Can we solve it in an approximate way?} \\
    \text{A: we can use least squares technique!}
\end{align*}
\]

Carl Friedrich Gauss
LS for Inhomogeneous Linear System

Let's consider a system of $p$ linear equations with $q$ unknowns

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \ldots + a_{1q}x_q &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \ldots + a_{2q}x_q &= b_2 \\
    \vdots \\
    a_{pq}x_1 + a_{p2}x_2 + \ldots + a_{pq}x_q &= b_p
\end{align*}
\]

\[\iff \quad Ax = b\]

We consider the case: $p > q$, and $\text{rank}(A) = q$

In general case, there is no solution!

Instead, we want to find a vector $x$ that minimizes the error:

\[
E(x) \equiv \sum_{i=1}^{p} \left( a_{i1}x_1 + \ldots + a_{iq}x_q - b_i \right)^2 = \left| Ax - b \right|^2
\]

Lin ZHANG, SSE, 2016
LS for Inhomogeneous Linear System

$$\mathbf{x}^* = \arg \min_{\mathbf{x}} E(\mathbf{x}) = \arg \min_{\mathbf{x}} \left\| \mathbf{A}\mathbf{x} - \mathbf{b} \right\|_2^2$$

$$\mathbf{x}^* = \left( \mathbf{A}^T \mathbf{A} \right)^{-1} \mathbf{A}^T \mathbf{b}$$

Pseudoinverse of $\mathbf{A}$

How about the pseudoinverse of $\mathbf{A}$ when $\mathbf{A}$ is square and non-singular?
Let's consider a system of $p$ linear equations with $q$ unknowns

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \ldots + a_{1q}x_q &= 0 \\
    a_{21}x_1 + a_{22}x_2 + \ldots + a_{2q}x_q &= 0 \\
    \quad \vdots \\
    a_{p1}x_1 + a_{p2}x_2 + \ldots + a_{pq}x_q &= 0
\end{align*}
\]

We consider the case: $p > q$, and $\text{rank}(A) = q$

Theoretically, there is only a trivial solution: $x = 0$

So, we add a constraint $\|x\|_2 = 1$ to avoid the trivial solution.
LS for Homogeneous Linear System

We want to minimize \( E(x) = \|Ax\|_2^2 \), subject to \( \|x\|_2 = 1 \)

\[ x^* = \arg \min_x E(x), \text{ s.t. } \|x\|_2 = 1 \quad (1) \]

Use the Lagrange multiplier to solve it,

\[ x^* = \arg \min_x \left[ \|Ax\|_2^2 + \lambda \left( 1 - \|x\|_2^2 \right) \right] \quad (2) \]

Solving the stationary point of the Lagrange function,

\[ \begin{cases} \\
\frac{\partial}{\partial x} \left[ \|Ax\|_2^2 + \lambda \left( 1 - \|x\|_2^2 \right) \right] = 0 \\
\frac{\partial}{\partial \lambda} \left[ \|Ax\|_2^2 + \lambda \left( 1 - \|x\|_2^2 \right) \right] = 0 \\
\end{cases} \quad (3) \]
LS for Homogeneous Linear System

\[
\frac{\partial}{\partial x} \left[ \|Ax\|_2^2 + \lambda \left( 1 - \|x\|_2^2 \right) \right] = 0 \tag{3}
\]

Then, we have

\[
A^T Ax = \lambda x
\]

\(x\) is the eigen-vector of \(A^T A\) associated with the eigenvalue \(\lambda\)

\[
E(x) = \|Ax\|_2^2 = x^T A^T Ax = x^T \lambda x = \lambda
\]

The unit vector \(x\) is the eigenvector associated with the minimum eigenvalue of \(A^T A\)
Contents

• Foundations of Projective Geometry
• Matrix Differentiation
• Lagrange Multiplier
• Least-squares for Linear Systems
• Homography Estimation
• What is Camera Calibration
• Single Camera Calibration
• Bird’s-eye-view Generation
Homography Estimation

Problem definition:
Given a set of points $\{x_i\}$ and a corresponding set of points $\{x'_i\}$ in a projective plane, compute the projective transformation that takes $x_i$ to $x'_i$.

We know there exists an $H$ satisfying $x'_i = Hx_i, i = 1, 2, ..., n$.

Coordinates of $\{x_i\}$ and $\{x'_i\}$ are known, we need to find $H$ where $H$ is a homography matrix.

$$H = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

It has 8 degrees of freedom.
Homography Estimation

4 point-correspondence pairs can uniquely determine a homography matrix since each correspondence pair solves two degrees of freedom.

\[
\begin{pmatrix}
  cu \\
  cv \\
  c
\end{pmatrix} = 
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix} \begin{pmatrix}
  x \\
  y \\
  1
\end{pmatrix} \rightarrow
\begin{cases}
  a_{11}x + a_{12}y + a_{13} = cu \\
  a_{21}x + a_{22}y + a_{23} = cv \\
  a_{31}x + a_{32}y + a_{33} = c
\end{cases}
\]

\[
\begin{cases}
  \frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{33}} = u \\
  \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + a_{33}} = v
\end{cases}
\]
Homography Estimation

4 point-correspondence pairs can uniquely determine a homography matrix since each correspondence pair solves two degrees of freedom.

\[
\begin{bmatrix}
-x - y - 1 & 0 & 0 & 0 & ux & uy & u \\
0 & 0 & 0 & -x & -y & -1 & vx & vy & v
\end{bmatrix}
\begin{bmatrix}
a_{11} \\
a_{12} \\
a_{13} \\
a_{21} \\
a_{22} \\
a_{23} \\
a_{31} \\
a_{32} \\
a_{33}
\end{bmatrix}
= 0
\]

Thus, four correspondence pairs generate 8 equations.
Homography Estimation

4 point-correspondence pairs can uniquely determine a homography matrix since each correspondence pair solves two degrees of freedom.

\[ A \mathbf{x} = 0 \quad (1) \]

Normally, \( \text{Rank}(A) = 8 \); thus (1) has 1 (9-8) solution vector in its solution space.
Homography Estimation

• How about the case when there are more than 4 correspondence pairs?
  • Use the LS method (for homogeneous case) to solve the model
Contents

• Foundations of Projective Geometry
• Matrix Differentiation
• Lagrange Multiplier
• Least-squares for Linear Systems
• Homography Estimation
• What is Camera Calibration
• Single Camera Calibration
• Bird’s-eye-view Generation
What is camera calibration?

- Camera calibration is a necessary step in 3D computer vision in order to extract metric information from 2D images
- It estimates the parameters of a lens and image sensor of the camera; you can use these parameters to correct for lens distortion, measure the size of an object in world units, or determine the location of the camera in the scene
- These tasks are used in applications such as machine vision to detect and measure objects. They are also used in robotics, for navigation systems, and 3-D scene reconstruction
What is camera calibration?

Before

After

Remove Lens Distortion

Estimate 3-D Structure from Camera Motion

Estimate Depth Using a Stereo Camera

Measure Planar Objects
What is camera calibration?

- Camera parameters include
  - Intrinsic parameters
  - Extrinsic parameters
  - Distortion coefficients
Contents

• Foundations of Projective Geometry
• Matrix Differentiation
• Lagrange Multiplier
• Least-squares for Linear Systems
• Homography Estimation
• What is Camera Calibration
• Single Camera Calibration
• Bird’s-eye-view Generation
Single Camera Calibration

- For simplicity, usually we use a pinhole camera model
Single Camera Calibration

- To model the image formation process, 4 coordinate systems are required
  - World coordinate system (3D space)
  - Camera coordinate system (3D space)
  - Retinal coordinate system (2D space)
  - Pixel coordinate system (2D space)
Single Camera Calibration

- To model the image formation process, 4 coordinate systems are required
Single Camera Calibration

- From the world CS to the camera CS

\[
\begin{bmatrix}
X_w, Y_w, Z_w
\end{bmatrix}^T \text{ is a 3D point represented in the WCS}
\]

In the camera CS, it is represented as,

\[
\begin{bmatrix}
X_c \\
Y_c \\
Z_c
\end{bmatrix} = R \begin{bmatrix}
X_w \\
Y_w \\
Z_w
\end{bmatrix} + t
\]

- a 3×3 rotation matrix (orthogonal)
- a 3×1 translation vector
Single Camera Calibration

- From the world CS to the camera CS

\[
\begin{bmatrix}
X_w, Y_w, Z_w
\end{bmatrix}^T \text{ is a 3D point represented in the WCS}
\]

In the camera CS, it is represented as,

\[
\begin{bmatrix}
X_c \\
Y_c \\
Z_c
\end{bmatrix}
= \begin{bmatrix}
X_w \\
Y_w \\
Z_w
\end{bmatrix}
+ \mathbf{t}
\]

Homogeneous form

\[
\begin{bmatrix}
X_c \\
Y_c \\
Z_c \\
1
\end{bmatrix}
= \begin{bmatrix}
\mathbf{R} & \mathbf{t} \\
\mathbf{0}^T & 1
\end{bmatrix}
\begin{bmatrix}
X_w \\
Y_w \\
Z_w \\
1
\end{bmatrix}
\]

(1)
Single Camera Calibration

• From the camera CS to the retinal CS

We can use a pin-hole model to represent the mapping from the camera CS to the retinal CS
Single Camera Calibration

- From the camera CS to the retinal CS

We can use a pin-hole model to represent the mapping from the camera CS to the retinal CS.

\[ P = [X_c, Y_c, Z_c]^T \]

*is a scene point while* \( P' = (x, y) \) *is its image on the retinal plane.*
Single Camera Calibration

• From the camera CS to the retinal CS

We can use a pin-hole model to represent the mapping from the camera CS to the retinal CS:

\[
\begin{bmatrix}
X_c \\
Y_c \\
Z_c \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
x \\
y \\
\end{bmatrix}
= \begin{bmatrix}
f \frac{X_c}{Z_c} \\
f \frac{Y_c}{Z_c} \\
\end{bmatrix}
\]

Homogeneous form:

\[
\begin{bmatrix}
x \\
y \\
1 \\
\end{bmatrix}
= \begin{bmatrix}
f 0 0 0 \\
0 f 0 0 \\
0 0 1 0 \\
\end{bmatrix}
\begin{bmatrix}
X_c \\
Y_c \\
Z_c \\
1 \\
\end{bmatrix}
\]

(2)
Single Camera Calibration

• From the retinal CS to the pixel CS

The unit for retinal CS \((x-y)\) is physical unit (e.g., mm, cm) while the unit for pixel CS \((u-v)\) is pixel.

One pixel represents \(dx\) physical units along the x-axis and represents \(dy\) physical units along the y-axis; the image of the optical center is \((u_0, v_0)\)

\[
\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & u_0 \\ \frac{1}{dx} & 0 & v_0 \\ 0 & \frac{1}{dy} & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}
\]
Single Camera Calibration

• From the retinal CS to the pixel CS

If the two axis of the pixel plane are not perpendicular to each other, another parameter $s$ is introduced to represent the skewness of the two axis

\[
\begin{bmatrix}
    u \\
    v \\
    1
\end{bmatrix} = \begin{bmatrix}
    \frac{1}{dx} & 0 & u_0 \\
    0 & \frac{1}{dy} & v_0 \\
    0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
    x \\
    y \\
    1
\end{bmatrix} = \begin{bmatrix}
    \frac{1}{dx} & s & u_0 \\
    0 & \frac{1}{dy} & v_0 \\
    0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
    x \\
    y \\
    1
\end{bmatrix} \tag{3}
\]
Single Camera Calibration

From Eqs.1~3, we can have

\[
Z_c \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{dx} & s & u_0 \\ 0 & \frac{1}{dy} & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_c \\ Y_c \\ Z_c \end{bmatrix} = \begin{bmatrix} \frac{f}{dx} & sf & u_0 & 0 \\ 0 & \frac{f}{dy} & v_0 & 0 \end{bmatrix} \begin{bmatrix} X_c \\ Y_c \\ Z_c \end{bmatrix}
\]

\[
\begin{bmatrix} \alpha & \gamma & u_0 & 0 \\ 0 & \beta & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_c \\ Y_c \\ Z_c \end{bmatrix} = \begin{bmatrix} \alpha & \gamma & u_0 & 0 \\ 0 & \beta & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \end{bmatrix} = \begin{bmatrix} X_w \\ Y_w \\ Z_w \end{bmatrix}
\]

Intrinsic Extrinsic
Single Camera Calibration

\[ Z_c \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = K \cdot [R \ t] \begin{bmatrix} X_w \\ Y_w \\ Z_w \end{bmatrix} \equiv \begin{bmatrix} \alpha & \gamma & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot [R \ t] \begin{bmatrix} X_w \\ Y_w \\ Z_w \end{bmatrix} \]

\((u_0, v_0)\), the coordinates of the principal point in the image plane

\(\alpha\) and \(\beta\), the scale factors in image \(u\) and \(v\) axes

\(\gamma\), describing the skewness of the two image axes

\(R\) and \(t\) determines the rigid transformation from the world coordinate system to the camera coordinate system

Altogether, there are 11 parameters to be determined
Single Camera Calibration

• To accurately represent an ideal camera, the camera model can include the radial and tangential lens distortion
  • Radial distortion occurs when light rays bend more near the edges of a lens than they do at its optical center; the smaller the lens, the greater the distortion
Single Camera Calibration

- To accurately represent an ideal camera, the camera model can include the radial and tangential lens distortion
  
  - Radial distortion occurs when light rays bend more near the edges of a lens than they do at its optical center; the smaller the lens, the greater the distortion

  \[
  u_{\text{distorted}} = u \left( 1 + k_1 r^2 + k_2 r^4 + k_3 r^6 \right)
  \]
  
  \[
  v_{\text{distorted}} = v \left( 1 + k_1 r^2 + k_2 r^4 + k_3 r^6 \right)
  \]

  where \( r^2 = u^2 + v^2 \)

  \( k_1, k_2, k_3 \) are the radial distortion coefficients of the lens
Single Camera Calibration

• To accurately represent an ideal camera, the camera model can include the radial and tangential lens distortion

  • Tangential distortion occurs when the lens and the image plane are not parallel
Single Camera Calibration

• To accurately represent an ideal camera, the camera model can include the radial and tangential lens distortion

  • Tangential distortion occurs when the lens and the image plane are not parallel

\[
\begin{align*}
  u_{\text{distorted}} &= u + \left( 2\rho_1 uv + \rho_2 \left( r^2 + 2u^2 \right) \right) \\
  v_{\text{distorted}} &= v + \left( 2\rho_2 uv + \rho_1 \left( r^2 + 2v^2 \right) \right)
\end{align*}
\]

\( \rho_1, \rho_2 \) are the tangential distortion coefficients of the lens
Single Camera Calibration

• The purpose of the camera calibration is to determine the values for the extrinsics, intrinsics, and distortion coefficients

• How to do?
  • Zhengyou Zhang’s method\cite{1} is a commonly used modern approach
  • A calibration board is needed; several images of the board need to be captured; based on the correspondence pairs (pixel coordinate and world coordinate of a feature point), equation systems can be obtained; by solving the equation systems, parameters can be determined

\cite{1} Z. Zhang, A flexible new technique for camera calibration, IEEE Trans. Pattern Analysis and Machine Intelligence, 2000
Single Camera Calibration

Calibration board
Single Camera Calibration

A set of Calibration board images (50~60)
Single Camera Calibration

• Matlab provides a “Camera Calibrator”
  • Straightforward to use
  • However, based on my experience, it is not as accurate as the routine provided in openCV3.0, especially for large FOV cameras (such as fisheye camera); thus, for some accuracy critical applications, I recommend to use the openCV function, though a little more complicated
  • It exports “cameraParams” as the calibration result
Single Camera Calibration
Single Camera Calibration

- For our purpose (measuring geometric metrics of a planar object), we use the camera parameters to undistort the image.
- The essence of this step is to make sure the transformation from a physical plane to the image plane can be represented by a linear projective matrix; or in other words, a straight line should be mapped to a straight line.
Single Camera Calibration

Original image

Undistorted image

Lin ZHANG, SSE, 2016
Contents

• Foundations of Projective Geometry
• Matrix Differentiation
• Lagrange Multiplier
• Least-squares for Linear Systems
• Homography Estimation
• What is Camera Calibration
• Single Camera Calibration
• Bird’s-eye-view Generation
Bird’s-eye-view Generation

• Our task is to measure the geometric properties of objects on a plane (e.g., conveyor belt)
• Such a problem can be solved if we have its bird-view image; bird’s-eye-view is easy for object detection and measurement
Bird’s-eye-view Generation

- Three coordinate systems are required
  - Bird’s-eye-view image coordinate system
  - World coordinate system
  - Undistorted image coordinate system

![Diagram showing the relationship between Bird’s-eye-view image, World Coordinate System (WCS), and Undistorted image]

Lin ZHANG, SSE, 2016
Bird’s-eye-view Generation

• Basic idea for bird’s-eye-view generation

Suppose that the transformation matrix from bird’s-eye-view to WCS is $P_{B \rightarrow W}$ and the transformation matrix from WCS to the undistorted image is $P_{W \rightarrow I}$

Then, given a position $(x_B, y_B, 1)^T$ on bird’s-eye-view, we can get its corresponding position in the undistorted image as

$$x_I = P_{W \rightarrow I} P_{B \rightarrow W} \begin{pmatrix} x_B \\ y_B \\ 1 \end{pmatrix}$$

Then, the intensity of the pixel $(x_B, y_B, 1)^T$ can be determined using some interpolation technique based on the neighborhood around $x_I$ on the undistorted image.
Bird’s-eye-view Generation

• Basic idea for bird’s-eye-view generation

Suppose that the transformation matrix from bird’s-eye-view to WCS is $P_{B \rightarrow W}$ and the transformation matrix from WCS to the undistorted image is $P_{W \rightarrow I}$

The key problem is how to obtain $P_{B \rightarrow W}$ and $P_{W \rightarrow I}$?
Bird’s-eye-view Generation

• Determine $P_{B \rightarrow W}$

\[
\begin{bmatrix}
-HN & H \\
2M' & -2
\end{bmatrix}
\]

\[
\begin{bmatrix}
HN & H \\
2M' & 2
\end{bmatrix}
\]

Lin ZHANG, SSE, 2016
• Determine $P_{B \rightarrow W}$

For a point $(x_B, y_B, 1)^T$ on bird’s-eye-view, the corresponding point on the world coordinate system is,

$$\begin{pmatrix} x_W \\ y_W \\ 1 \end{pmatrix} = \begin{bmatrix} \frac{H}{M} & 0 & -\frac{HN}{2M} \\ 0 & -\frac{H}{M} & \frac{H}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_B \\ y_B \\ 1 \end{pmatrix} \equiv P_{B \rightarrow W} \begin{pmatrix} x_B \\ y_B \\ 1 \end{pmatrix}$$

Please verify!!
**Bird’s-eye-view Generation**

- Determine $P_{W \rightarrow I}$

The physical plane (in WCS) and the undistorted image plane can be linked via a homography matrix $P_{W \rightarrow I}$

$$x_I = P_{W \rightarrow I} x_W$$

If we know a set of correspondence pairs $\{x_{ii}, x_{Wi}\}_{i=1}^N$, $P_{W \rightarrow I}$ can be estimated using the least-square method.
Bird’s-eye-view Generation

• Determine $P_{W \rightarrow I}$

A set of point correspondence pairs; for each pair, we know its coordinate on the undistorted image plane and its coordinate in the WCS
When $P_{B\rightarrow W}$ and $P_{W\rightarrow I}$ are known, the bird’s-eye-view can be generated via,

$$x_I = P_{W\rightarrow I}P_{B\rightarrow W} \begin{pmatrix} x_B \\ y_B \\ 1 \end{pmatrix} \equiv P_{B\rightarrow I} \begin{pmatrix} x_B \\ y_B \\ 1 \end{pmatrix}$$
Bird’s-eye-view Generation

Original image

Bird’s-eye-view
Bird-view Generation

Another example

Original fish-eye image

Undistorted image
Bird-view Generation

Another example

Original fish-eye image

Bird’s-eye-view
Thanks for your attention