Question 2

1-2 (Math) Gaussian function is

\[ G(x, y; \sigma) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) \]

The scale-normalized Laplacian of Gaussian (LOG) is

\[ \text{LoG} = \sigma^2 \nabla^2 G \]

Please verify that Difference of Gaussian (DOG)

\[ \text{DoG} = G(x, y; k\sigma) - G(x, y; \sigma) \]

can be a good approximation of LoG.

Answer.

\[ \frac{\partial G}{\partial x} = -\frac{x}{2\pi\sigma^4} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) \]
\[ \frac{\partial G}{\partial y} = -\frac{y}{2\pi\sigma^4} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) \]

\[ \frac{\partial^2 G}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial G}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial G}{\partial y}\right) = \frac{x^2 - \sigma^2}{2\pi\sigma^6} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) \]
\[ \frac{\partial^2 G}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial G}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial G}{\partial y}\right) = \frac{y^2 - \sigma^2}{2\pi\sigma^6} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) \]

\[ \nabla^2 G = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = \frac{x^2 + y^2 - 2\sigma^2}{2\pi\sigma^6} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) \]

\[ \text{LoG} = \sigma^2 \nabla^2 G = \frac{x^2 + y^2 - 2\sigma^2}{2\pi\sigma^4} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) \]

\[ \frac{\partial G}{\partial \sigma} = -\frac{1}{\pi\sigma^3} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) + \frac{1}{2\pi\sigma^5} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) \left(x^2 + y^2\right) \]
\[ = \frac{x^2 + y^2 - 2\sigma^2}{2\pi\sigma^5} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) \]
\[ = \sigma \nabla^2 G \quad (1) \]

Hence,

\[ \frac{\partial G}{\partial \sigma} = \sigma \nabla^2 G \]

Performing the calculus of difference on the equation (1) yields,

\[ \sigma \nabla^2 G = \frac{\partial G}{\partial \sigma} \approx \frac{G(x, y, k\sigma) - G(x, y, \sigma)}{k\sigma - \sigma} \]
DoG = G(x, y, k\sigma) - G(x, y, \sigma) \approx (k - 1)\sigma^2 \nabla^2 G = (k - 1) \text{LoG}

As the constant term \((k - 1)\) does not affect the location of the extrema. So DoG is an approximation of \(\sigma^2 \nabla^2 G\), Q.E.D.